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Three-dimensional quantum algebras: a Cartan-like point of view

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Abstract

A perturbative quantization procedure for Lie bialgebras is introduced. The relevance of the choice of a completely symmetrized basis of the quantum universal enveloping algebra is stressed. Sets of elements of the quantum algebra that play a role similar to generators in the case of Lie algebras are considered and a Cartan-like procedure applied to find a representative for each class of quantum algebras. The method is used to construct and classify all three-dimensional complex quantum algebras that are compatible with a given type of coproduct. The quantization of all Lie algebras that, in the classical limit, belong to the most relevant sector in the classification for three-dimensional Lie bialgebras given in [1] is thus performed. New quantizations of solvable algebras, whose simplicity makes them suitable for possible physical applications, are obtained and already known related quantum algebras recovered.

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1. Introduction

The Cartan classification of semisimple Lie algebras has certainly facilitated some of their applications in physics. However, the existence of the analogous classification scheme for quantum algebras is still an open problem, and the physical applications of quantum deformations are also far from being fully developed and understood. For these reasons, this paper deals with some aspects of the problem of the construction and classification of quantum universal enveloping algebras (hereafter, quantum algebras) [2–5].

It is well known that any quantum algebra $(U_z(\mathfrak{a}), \Delta)$ with deformation parameter z defines a unique Lie bialgebra structure $(\mathfrak{a}, \eta)$, a pair determined by the Lie algebra $\mathfrak{a}$ and a skew-symmetric linear map (cocommutator) $\eta : \mathfrak{a} \to \mathfrak{a} \otimes \mathfrak{a}$. Such a cocommutator η is defined as the first-order skew symmetric part of the coproduct

$$\eta(X) = \tfrac{1}{2}(\Delta(X) - \sigma \circ \Delta(X)) \qquad \mathrm{mod}\, z^2 \quad \forall X \in \mathfrak{a} \tag{1.1}$$

where σ is the flip operator $\sigma(X \otimes Y) = Y \otimes X$.

Therefore, quantum deformations of a given Lie algebra $\mathfrak{a}$ can be classified according to this 'semi classical' limit. Moreover, we recall that Lie bialgebras are in one to one correspondence with Poisson–Lie structures on the group $Lie(\mathfrak{a})$ [6] that arise again as the first order in z of the quantum groups dual to the quantum algebras $(U_z(\mathfrak{a}), \Delta)$. With this in mind, some classifications of Lie bialgebra structures for several physically relevant Lie algebras have been obtained. For the three-dimensional (3D) case we will recall the full classification of Lie bialgebras given in [1] and for some higher dimensional Lie bialgebras we refer to [7] and references therein.

However, it is clear that the inverse problem has also to be faced: that is, to obtain general recipes for the 'quantization' (i.e. the associated quantum algebra) of a given Lie bialgebra $(\mathfrak{a}, \eta)$. Although existence of such an object is indeed guaranteed (see [3], chapter 6), only for coboundary triangular bialgebras the Drinfel'd twist operator gives rise to the associated quantum algebra [4]. However, quasi-triangular and even non-coboundary Lie bialgebras do exist and for them the twist operator approach to quantization is not available. We recall that some attempts in order to get structural properties of the quantization of arbitrary Lie bialgebras have been performed (see [8, 9] for a prescription to get the quantum coproduct—but not the deformed commutation rules—for a wide class of examples). Moreover, to our knowledge, a general investigation concerning the uniqueness of this Lie bialgebra quantization process has not been yet given and only restrictive results for certain deformations of simple Lie algebras have been obtained (see [10], chapter 11). As a consequence of the above-mentioned facts, a complete classification of quantum algebras in the spirit of the Cartan classification for Lie algebras is still far to be reached.

In this paper we present a 'direct approach' to the quantization problem, together with an operational procedure *à la Cartan* (in a sense that will be made more explicit in the sequel) for the classification of quantum algebras. The essential ingredients of the quantization method are described in section 2. In particular, given a Lie bialgebra we shall firstly obtain a coassociative quantum coproduct with some outstanding symmetry properties and, afterwards, we shall solve order by order the compatibility equations for the deformed commutation rules.

In section 3 this approach will be explicitly developed for a relevant type of cocommutator η that is compatible with all the non-isomorphic 3D complex Lie algebras, as described—for instance—in Jacobson [11]. This cocommutator gives rise to a very simple coproduct and to deformed commutators that are manageable enough. In this way we obtain three different families of quantum algebras whose deformed commutation rules are quite general and depend on several structure constants. We would like to stress that, instead the usual Poincaré–Birkhoff–Witt (PBW) basis, our quantization procedure makes use of a completely symmetrized basis in the universal enveloping algebra. This choice has not been previously considered in the literature, but is certainly convenient in quantum physics and leads to a straightforward classical (Poisson) limit for all the quantum algebras here presented.

In order to classify such quantizations, in section 4 we consider the equivalence transformations that are defined through invertible maps in $U_z(\mathfrak{a})$ that preserve a form-invariant coproduct for the basis elements of the algebra. This constraint regarding the form invariance of the coproduct can be understood as a way to identify certain basis elements as some sort of 'generators' of a quantum algebra. In other words, we follow for quantum algebras the 'Cartan' approach to the classification of Lie algebras within $U(\mathfrak{a})$, where only linear transformations in the space of generators leaving invariant their (primitive) coproduct are performed. After such a classification is systematically developed, new quantizations of 3D solvable algebras are obtained, and already known results are recovered. The above-mentioned simplicity of the algebraic and coalgebraic structures of these new quantizations notably increases their possible interest in physical applications. In order to make more precise the range of 3D deformations that have been covered, a detailed comparison with the complete classification of 3D real Lie bialgebra structures presented in [1] is explicitly given in the final section, together with some further comments and conclusions.

2. The quantization method

Let us consider the Lie algebra $\mathfrak{a}$ and its universal enveloping algebra $U(\mathfrak{a})$ [11]. If we define the coproduct: Δ_0, counit: ϵ and antipode: γ ($\forall X \in \mathfrak{a}$)

$$\Delta_0(X) = 1 \otimes X + X \otimes 1 \qquad \Delta_0(1) = 1 \otimes 1 \quad \epsilon(X) = 0 \quad \epsilon(1) = 1 \quad \gamma(X) = -X$$

and we extend by linearity all these (anti)automorphisms to the full $U(\mathfrak{a})$; we shall endow the universal enveloping algebra with a Hopf algebra structure. In general, an element Y of a Hopf algebra is called *primitive* if

$$\Delta(Y) = 1 \otimes Y + Y \otimes 1.$$

It can be shown that the only primitive elements of $U(\mathfrak{a})$ under the coproduct Δ_0 are the generators of $\mathfrak{a}$ (this result is known as the Friedrichs theorem [12]). Note that an essential property of the Hopf algebra $U(\mathfrak{a})$ is its cocommutativity, since Δ_0 is invariant under the action of the flip operator σ.

Let us now consider the Hopf algebra $U(\mathfrak{a})$ and a deformation parameter z. A quantum algebra $(U_z(\mathfrak{a}), \Delta)$ is the Hopf algebra of formal power series in the deformation parameter z with coefficients in $U(\mathfrak{a})$ and such that [3]

$$U_z(\mathfrak{a})/zU_z(\mathfrak{a}) \simeq U(\mathfrak{a}).$$

Therefore, $U(\mathfrak{a})$ is obtained (as the Hopf algebra) in the limit $z \to 0$, and the first order in z of the coproduct is directly related to the cocommutator of an underlying Lie bialgebra $(\mathfrak{a}, \eta)$ through (1.1). In this way, Lie bialgebras can be used to characterize quantum deformations. Amongst all the Hopf algebra axioms to be imposed, we recall that the quantum coproduct Δ has to be a coassociative map, namely

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \tag{2.1}$$

From now on, we shall refer to the 'quantization' of a given Lie bialgebra $(\mathfrak{a}, \eta)$ as the problem of finding a quantum algebra $(U_z(\mathfrak{a}), \Delta)$ such that (1.1) is fulfilled (we recall that the uniqueness of such a construction cannot be taken for granted).

The general quantization procedure that we propose is based on three essential ingredients that can be considered whatever the dimension of the Lie bialgebra is. The first one is a generalized cocommutativity property for the coproduct, the second one is related to the choice of a basis in the universal enveloping algebra and the third one selects a given type of power series expansion for the deformed commutation rules.

(1) *Generalized cocommutativity*. In general, we impose the (noncocommutative) quantum coproduct Δ to be invariant under the composition $\tilde{\sigma} = \sigma \circ T$ of the flip operator σ and a change of sign of (all) the deformation parameter(s):

$$\tilde{\sigma} \circ \Delta = \Delta \qquad \text{where} \qquad \tilde{\sigma} = T \circ \sigma \qquad \text{and} \qquad T(z) = -z.$$

We point out that definition (1.1) for the underlying cocommutator implies that the deformation parameters do appear explicitly as multiplicative factors within the cocommutator. This symmetry property of the coproduct can be imposed in any dimension and makes much easier the procedures of symmetrization and 'Hermitation' [13]. In particular, given a certain Lie bialgebra $(\mathfrak{a}, \eta)$, the above assumption implies that the first-order deformation of the coproduct will be just given by the (skew symmetric) Lie bialgebra cocommutator

$$\Delta(X) = \Delta_0(X) + \eta(X) + O[z^2] \qquad X \in \mathfrak{g}. \tag{2.2}$$

Moreover, the invariance of Δ under the transformation $\tilde{\sigma}$ together with the fact that the coproduct is an algebra homomorphism with respect to the deformed commutation rules $[\cdot,\cdot]_z$ (i.e., $\Delta([\cdot,\cdot]_z) = [\Delta(\cdot), \Delta(\cdot)]_z$) implies that $[\cdot,\cdot]_z$ has to be an even function in the deformation parameter z.

(2) *The choice of a basis in* $U_z(\mathfrak{a})$. In contradistinction with previous works on this subject in which the PBW basis $X_1^{\alpha} X_2^{\beta} \ldots X_l^{\zeta}$ is considered, we introduce a basis in $U_z(\mathfrak{a})$ given by the completely symmetrized monomials. Thus, we define the linear operator Sym by

$$\mathrm{Sym}\left\{\sum c_{\alpha\beta\ldots\zeta} X_1^{\alpha} X_2^{\beta} \ldots X_l^{\zeta}\right\} = \sum c_{\alpha\beta\ldots\zeta}\, \mathrm{Sym}\left\{X_1^{\alpha} X_2^{\beta} \ldots X_l^{\zeta}\right\}$$

where

$$\mathrm{Sym}\{A_1 \ldots A_n\} := \frac{1}{n!} \sum_{p \in \mathrm{S_n}} p(A_1 \ldots A_n),$$

with S_n the permutation group of n elements. Note that Sym is just the identity operator for commutative basis elements.

This symmetrization procedure, although very convenient in quantum mechanical terms, has not been previously considered in the literature, and turns out to be very efficient in order to get the explicit form of the deformed commutation rules. We remark that one of the main advantages of the symmetrized basis in the quantum algebra is the fact that if we replace the deformed commutation rules by Poisson brackets, the corresponding Poisson–Hopf algebra is uniquely (and immediately) defined.

(3) *Deformed commutation rules*. We will assume

$$[X, Y]_z = \frac{1}{z} \mathrm{Sym}(f(zX_1, zX_2, \ldots, zX_l)) \tag{2.3}$$

where f is a meromorphic function at $z = 0$ and also odd in z.

The discussion can be extended to the multiparametric case considering meromorphic deformations of Lie algebras with $\rho_{ij} = z_i/z_j$ fixed.

Under all these assumptions, given a Lie bialgebra $(\mathfrak{a}, \eta)$, the 'direct' quantization procedure that we propose would be sketched as follows:

- Assume that the first-order coproduct is of the form (2.2).
- Order by order in the deformation parameter(s), get the relations coming from the coassociativity constraint (2.1) and solve them recursively by taking into account the invariance under $\tilde{\sigma}$ of the solution, thus obtaining the full quantum coproduct.
- Obtain, again order by order, the deformed commutation rules by solving the compatibility equations coming from the fact that the coproduct has to be an algebra homomorphism.

3. Three-dimensional quantum algebras

Let $\{A, B, C\}$ be the generators of an arbitrary complex 3D Lie algebra $\mathcal{L}$ with commutators

$$\begin{aligned}
[A, B] &= c_1 A + c_2 B + c_3 C \\
[A, C] &= b_1 A + b_2 B + b_3 C \\
[B, C] &= a_1 A + a_2 B + a_3 C
\end{aligned} \tag{3.1}$$

where the structure constants are complex numbers subjected to some (nonlinear) relations coming from the Jacobi identity.

We recall that the complete classification of the 3D complex Lie algebras is given in [11] (see, for instance, [14] for the real case). According to the dimension of the derived algebra $\mathcal{L}' = [\mathcal{L}, \mathcal{L}]$, the nonisomorphic classes of 3D Lie algebras read

- *Type I*. $\dim \mathcal{L}' = 0$. Then $\mathcal{L}$ is Abelian. We shall not consider this case from the point of view of quantum deformations, since any coassociative coalgebra with primitive non-deformed limit is compatible with the Abelian commutation rules.
- *Type II*. $\dim \mathcal{L}' = 1$. We have two algebras, the Heisenberg–Weyl algebra and a central extension $\mathcal{L} = \mathcal{B} \oplus \mathcal{C}$ of the Borel algebra, where $\mathcal{B}$ is a Borel algebra and $\mathcal{C}$ commutes with $\mathcal{B}$.
- *Type III*. $\dim \mathcal{L}' = 2$. We have the family on non-isomorphic Lie algebras labelled by the nonzero complex number α with commutators

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = X_1 \qquad [X_2, X_3] = \alpha X_2 \tag{3.2}$$

  and the Lie algebra

$$[X_1, X_2] = 0 \qquad [X_1, X_3] = X_1 + X_2 \qquad [X_2, X_3] = X_2. \tag{3.3}$$

  Note that the algebra with $\alpha = 1$ is the solvable algebra generated by a dilation X_3 and two translations $\{X_1, X_2\}$. The case $\alpha = -1$ corresponds to the (complex) Euclidean Lie algebra.
- *Type IV*. $\dim \mathcal{L}' = 3$. The only element in this class is the simple Lie algebra $\mathcal{A}_1$.

Obviously, once a set of values for the structure constants $\{a_i, b_i, c_i\}$ is given, a suitable linear transformation $X_i = X_i(A, B, C)$ with complex coefficients can be found in such a way that the Lie algebra (3.1) is reduced to one of the Jacobson cases. However, in (3.1) there is a symmetry in $\{A, B, C\}$ that can be broken in the quantization.

In particular, by looking at the complexified form of the classification of 3D Lie bialgebras given in [1], one can realize that a cocommutator of the form

$$\eta(A) = 0 \qquad \eta(B) = zA \wedge B \qquad \eta(C) = \rho z A \wedge C \qquad z, \rho \in \mathbb{C} \tag{3.4}$$

defines a Lie bialgebra structure for each of the types of Lie algebras given in Jacobson's classification provided that different linear transformations $X_i = X_i(A, B, C)$ are defined and that ρ takes certain appropriate values. Since we are interested in obtaining the most general types of deformed commutation rules arising in 3D quantizations, we shall apply the perturbative quantization procedure described in the previous section in order to get a quantum coproduct coming from the cocommutator (3.4) together with a compatible deformation of the commutation rules (3.1). We remark that the cocommutator (3.4) can be thought of as a two parameter structure in z and $\chi = \rho z$. Moreover, complex values for all the parameters (including the structure constants $\{a_i, b_i, c_i\}$ and ρ) will be considered, and the results here presented will also contain the quantizations at the roots of unity.

Let us follow step by step the procedure introduced in section 2. Firstly, we assume that the quantum coproduct will be of the form (2.2), namely

$$\begin{aligned}\Delta(A) &= 1\otimes A + A\otimes 1\\ \Delta(B) &= 1\otimes B + B\otimes 1 + zA\wedge B + O[z^2]\\ \Delta(C) &= 1\otimes C + C\otimes 1 + \rho z A\wedge C + O[z^2].\end{aligned} \tag{3.5}$$

In this case, it is straightforward to prove that the following coassociative coproduct has a first order given by the cocommutators (3.4) and is invariant under the transformation $\tilde{\sigma}$

$$\begin{aligned}\Delta(A) &= 1\otimes A + A\otimes 1\\ \Delta(B) &= \mathrm{e}^{zA}\otimes B + B\otimes \mathrm{e}^{-zA}\\ \Delta(C) &= \mathrm{e}^{\rho zA}\otimes C + C\otimes \mathrm{e}^{-\rho zA}.\end{aligned} \tag{3.6}$$

Now we have to obtain the deformed commutation rules by solving the compatibility equations coming from the fact that the coproduct has to be an algebra homomorphism. Since the $\tilde{\sigma}$-invariance of the coproduct implies that the first-deformed term in the commutation rules has to be of order z^2, the first-order coproduct (3.5) has to be compatible with the non-deformed commutation rules. This condition leads to the following relations between the structure constants and ρ:

$$b_2(1-\rho)=0 \qquad c_3(1-\rho)=0 \qquad b_1=-\rho a_2 \qquad a_3=\rho c_1 \qquad b_3=-\rho c_2. \tag{3.7}$$

Jacobi identity imposes the further relation

$$(1-\rho)[a_2c_1(1+\rho)-a_1c_2]=0. \tag{3.8}$$

Now we can distinguish the following three cases:

(1) *Case* $\rho\neq\pm1$:

$$\begin{aligned}&b_2=0 \qquad c_3=0 \qquad b_1=-\rho a_2 \qquad a_3=\rho c_1 \qquad b_3=-\rho c_2\\ &a_2c_1(1+\rho)-a_1c_2=0.\end{aligned} \tag{3.9}$$

Note that the space of parameters is 3D.

(2) *Case* $\rho=+1$:

$$b_1=-a_2 \qquad a_3=c_1 \qquad b_3=-c_2. \tag{3.10}$$

In this case the space of parameters is 6D.

(3) *Case* $\rho=-1$:

$$b_2=0 \qquad c_3=0 \qquad b_1=a_2 \qquad a_3=-c_1 \qquad b_3=c_2 \qquad a_1c_2=0. \tag{3.11}$$

Here, the space of parameters is 3D.

In all these cases, when the quantization procedure is extended to higher orders no additional conditions between the structure constants are found, in spite of the fact that we have assumed no dependence of the structure constants on the deformation parameter.

Now, we are in conditions to obtain the deformed commutators. So, the integration of the above equations to all orders gives the most general families of 3D quantum algebras that are compatible with the deformed coproduct (3.6). We find

(1) *Case* $\rho \neq \pm 1$:

(1.1) $c_2 \neq 0$:

$$
\begin{aligned}
&[A, B] = c_1 \sinh(zA)/z + c_2 B \\
&[A, C] = -a_2 \sinh(z\rho A)/z - \rho c_2 C \\
&[B, C] = \frac{a_2 c_1}{c_2} \frac{\sinh[z(1+\rho)A]}{z} + a_2 \mathrm{Sym}[B \cosh(z\rho A)] + \rho c_1 \mathrm{Sym}[C \cosh(zA)].
\end{aligned}
\tag{3.12}
$$

(1.2) $c_2 = 0, a_2 = 0$:

$$
\begin{aligned}
&[A, B] = c_1 \sinh(zA)/z \\
&[A, C] = 0 \\
&[B, C] = a_1 \frac{\sinh[z(1+\rho)A]}{z(1+\rho)} + \rho c_1 C \cosh(zA).
\end{aligned}
\tag{3.13}
$$

Note that the case with $(c_1 = 0, c_2 = 0)$ is equivalent to the one with $(c_2 = 0, a_2 = 0)$.

(2) *Case* $\rho = +1$:

(2.1)

$$
\begin{aligned}
&[A, B] = c_1 \sinh(zA)/z + c_2 B + c_3 C \\
&[A, C] = -a_2 \sinh(zA)/z + b_2 B - c_2 C \\
&[B, C] = a_1 \sinh(2zA)/(2z) + \mathrm{Sym}\{(a_2 B + c_1 C) \cosh(zA)\}
\end{aligned}
\tag{3.14}
$$

(3) Case $\rho = -1$:

(3.1) $c_2 \neq 0$:

$$
\begin{aligned}
&[A, B] = c_1 \sinh(zA)/z + c_2 B \\
&[A, C] = b_1 \sinh(zA)/z + c_2 C \\
&[B, C] = \mathrm{Sym}\{(b_1 B - c_1 C) \cosh(zA)\}.
\end{aligned}
\tag{3.15}
$$

(3.2) $c_2 = 0$:

$$
\begin{aligned}
&[A, B] = c_1 \sinh(zA)/z \\
&[A, C] = b_1 \sinh(zA)/z \\
&[B, C] = a_1 A + \mathrm{Sym}\{(b_1 B - c_1 C) \cosh(zA)\}.
\end{aligned}
\tag{3.16}
$$

It is worth noticing that the symmetric role played by both non-primitive generators in the quantization procedure is reflected by a symmetry in many of the above-reported algebras.

4. Equivalence and classification

In general, two quantum algebras are said to be equivalent (isomorphic) if there exists an invertible (nonlinear in many cases) map between their corresponding quantum universal enveloping algebras as Hopf algebras. In this way, equivalence classes of quantum deformations can be defined. But it is also clear that, due to the infinite number of possibilities given by arbitrary nonlinear transformation maps, such equivalence classes are huge. For classification purposes some general criteria is, thus, needed in order to perform an appropriate choice of a 'canonical' representative within each equivalence class. Also, it seems quite natural to assume that such a choice has to be related with some structural property regarding the coproduct. Moreover, the definition of such canonical representatives would lead to the objects that can be properly called the 'generators' of the quantum algebra.

In fact, the standard classification of (non-deformed) Lie algebras can be understood as a two step operation: first from the universal enveloping algebras the generators are extracted as the elements with the simplest coproduct (primitive, in agreement with the Friedrichs theorem); then the (Cartan) classification of Lie algebras is performed choosing between the generators a basis having the 'simplest' commutation rules (minimum number of non-vanishing structure constants). We stress that, in this last step, only linear transformations are allowed because they leave the coproduct invariant in form (primitive, as we have not deformations).

From this perspective, we propose a definition of the 'canonical' representatives of quantum algebras by following a similar procedure. In this case it is essential to realize that a quantum algebra $U_z(\mathfrak{a})$ is endowed with a deformed coproduct Δ which is no longer cocommutative. Thus, in order to find the 'canonical' generators, we shall start from (3.6), by inspection the simplest coproduct compatible with (3.5). Then we shall move within the equivalence subclass defined through the restricted set of Hopf algebra isomorphisms of quantum universal enveloping algebras that preserve a form-invariant coproduct and through such restricted isomorphisms we shall look for representatives with 'irreducible' deformed commutation rules having a minimal number of non-zero terms.

By proceeding in this way, we have succeeded in classifying all the non-isomorphic quantum algebras that are contained in the three multiparameter families given in the previous section. Let us explicitly obtain them by eliminating many irrelevant parameters through mappings preserving a form-invariant coproduct.

4.1. Case 1: $\rho \neq \pm 1$

Let us start with the case $\rho \neq \pm 1$. Let us consider the following transformation [15]:

$$\begin{aligned} &\mathcal{A} = \alpha A \qquad \mathcal{B} = \beta B + \delta \frac{\sinh(A)}{z} \qquad \mathcal{C} = \nu C + \eta \frac{\sinh(z\rho A)}{z\rho} \\ &\hat{z} = \alpha^{-1} z \qquad \alpha, \beta, \delta, \nu, \eta \in \mathbb{C}. \end{aligned} \tag{4.1}$$

After this transformation the coproduct (3.6) becomes

$$\begin{aligned} &\Delta(\mathcal{A}) = 1 \otimes \mathcal{A} + \mathcal{A} \otimes 1 \\ &\Delta(\mathcal{B}) = \mathrm{e}^{\hat{z}\mathcal{A}} \otimes \mathcal{B} + \mathcal{B} \otimes \mathrm{e}^{-\hat{z}\mathcal{A}} \\ &\Delta(\mathcal{C}) = \mathrm{e}^{\hat{z}\rho\mathcal{A}} \otimes \mathcal{C} + \mathcal{C} \otimes \mathrm{e}^{-\hat{z}\rho\mathcal{A}} \end{aligned} \tag{4.2}$$

i.e. it is form-invariant. So, we can say that all the elements related through (4.1) belong of the same class of quantum algebras. In particular, we get the following possibilities:

(1.1) $c_2 \neq 0$. In this case the above-mentioned change of basis (4.1) can be reduced to

$$\begin{aligned} &\mathcal{A} = A/c_2 \qquad \mathcal{B} = c_2 B + c_1 \frac{\sinh(zA)}{z} \qquad \mathcal{C} = c_2 C + a_2 \frac{\sinh(z\rho A)}{z\rho} \\ &\hat{z} = c_2 z. \end{aligned} \tag{4.3}$$

(1.1.1) Under this change of basis we obtain the new Lie commutators

$$[\mathcal{A}, \mathcal{B}] = \mathcal{B} \qquad [\mathcal{A}, \mathcal{C}] = -\rho\mathcal{C} \qquad [\mathcal{B}, \mathcal{C}] = 0 \tag{4.4}$$

which correspond to a quantization of the Lie algebra (3.2). Its associated bialgebra is a non-coboundary one, i.e. there is no classical r-matrix.

(1.2) $c_2 = 0$. We have two quantum algebras. Following a procedure analogous to (4.3) they can be written as

(1.2.1) $a_2 = 0, c_1 \neq 0$

$$[\mathcal{A}, \mathcal{B}] = \sinh(\hat{z}\mathcal{A})/\hat{z} \qquad [\mathcal{A}, \mathcal{C}] = 0 \qquad [\mathcal{B}, \mathcal{C}] = \rho\mathcal{C}\cosh(\hat{z}\mathcal{A}). \tag{4.5}$$

The underlying Lie bialgebra is a coboundary one: the classical r-matrix is $r = \hat{z}\mathcal{A} \wedge \mathcal{B}$. It is non-standard, i.e. it verifies the classical Yang–Baxter equation. This is another quantum deformation of the Lie algebra (3.2).

(1.2.2) $a_2 = 0, a_1 \neq 0, c_1 = 0$

$$[\mathcal{A}, \mathcal{B}] = 0 \qquad [\mathcal{A}, \mathcal{C}] = 0 \qquad [\mathcal{B}, \mathcal{C}] = \frac{\sinh[\hat{z}(1+\rho)\mathcal{A}]}{\hat{z}(1+\rho)}. \quad (4.6)$$

This is also a coboundary deformation with standard r-matrix given by $r = \hat{z}\mathcal{B} \wedge \mathcal{C}$, i.e. r fulfils the modified classical Yang–Baxter equation.

Note that for $\rho = 0$ we have obtained two deformations of the extended Borel algebra and one deformation of the Heisenberg–Weyl algebra (both Lie algebras belong to type II in Jacobson's classification).

4.2. *Case 2:* $\rho = +1$

The equivalence classes are defined by applying to (3.14) the following transformation:

$$\begin{aligned} &\mathcal{A} = \alpha A \qquad \mathcal{B} = \beta B + \gamma C + \delta \frac{\sinh(zA)}{z} \qquad \mathcal{C} = \mu B + \nu C + \eta \frac{\sinh(zA)}{z} \\ &\hat{z} = \alpha^{-1} z \qquad \alpha, \beta, \gamma, \delta, \mu, \nu, \eta \in \mathbb{C}. \end{aligned} \quad (4.7)$$

This transformation allows us to distinguish the quantum algebras characterized by $b_2c_3 + {c_2}^2 \neq 0$ and those in which $b_2c_3 + {c_2}^2 = 0$.

(2.1) $b_2c_3 + {c_2}^2 \neq 0$

(2.1.1)

$$[\mathcal{A}, \mathcal{B}] = \mathcal{B} \qquad [\mathcal{A}, \mathcal{C}] = -\mathcal{C} \qquad [\mathcal{B}, \mathcal{C}] = \frac{\sinh(2\hat{z}\mathcal{A})}{2\hat{z}}. \quad (4.8)$$

This quantum algebra is just $\mathcal{A}_1(q)$. The classical r-matrix is $r = z\mathcal{B} \wedge \mathcal{C}$, and is a standard one.

(2.1.2)

$$[\mathcal{A}, \mathcal{B}] = \mathcal{B} \qquad [\mathcal{A}, \mathcal{C}] = -\mathcal{C} \qquad [\mathcal{B}, \mathcal{C}] = 0. \quad (4.9)$$

This is the complexification of the quantum two-dimensional Euclidean algebra $\mathcal{E}(2)$ that was firstly obtained as the contraction of the above $su_q(2)$ deformation in [16]. It is a non-coboundary one.

(2.2) $b_2c_3 + {c_2}^2 = 0$

(2.2.1) $a_2c_2 + b_2c_1 \neq 0$

$$[\mathcal{A}, \mathcal{B}] = -\frac{\sinh(\hat{z}\mathcal{A})}{\hat{z}} \qquad [\mathcal{A}, \mathcal{C}] = \mathcal{B} \qquad [\mathcal{B}, \mathcal{C}] = -\mathrm{Sym}\{\mathcal{C}\cosh(\hat{z}\mathcal{A})\}. \quad (4.10)$$

This is the symmetrized version of the well-known Jordanian deformation of $\mathcal{A}(1)$ [17] with non-standard classical r-matrix $r = \hat{z}\mathcal{A} \wedge \mathcal{B}$. According to the commutation relations (4.10) we obtain that

$$\mathrm{Sym}(\mathcal{C}\cosh(\hat{z}\mathcal{A})) = \frac{1}{2}(\mathcal{C}\cosh(\hat{z}\mathcal{A}) + \cosh(\hat{z}\mathcal{A})\mathcal{C})) + \frac{1}{12}\hat{z}^2 \sinh\frac{2\hat{z}\mathcal{A}}{2\hat{z}}.$$

Thus, the quantum algebra presented in [17] is a particular case of (3.14) with z-dependent parameters.

(2.2.2) $a_2c_2 + b_2c_1 = 0$. We obtain the following algebras:

(2.2.2.1)

$$[\mathcal{A}, \mathcal{B}] = \frac{\sinh(\hat{z}\mathcal{A})}{\hat{z}} \qquad [\mathcal{A}, \mathcal{C}] = 0 \qquad [\mathcal{B}, \mathcal{C}] = \mathcal{C}\cosh(\hat{z}\mathcal{A}). \quad (4.11)$$

We recover a non-standard deformation of the Euclidean group in two dimensions $E(2)$ [18]. The classical r-matrix is non-standard, $r = \hat{z}\mathcal{A} \wedge \mathcal{B}$.

(2.2.2.2)

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=-\mathcal{B} \qquad [\mathcal{B},\mathcal{C}]=\frac{\sinh(2\hat{z}\mathcal{A})}{2\hat{z}}. \tag{4.12}$$

We have the standard deformation of $E(2)$ with classical r-matrix $r=\hat{z}\mathcal{B}\wedge\mathcal{C}$.

(2.2.2.3)

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=0 \qquad [\mathcal{B},\mathcal{C}]=\frac{\sinh(2\hat{z}\mathcal{A})}{2\hat{z}}. \tag{4.13}$$

It corresponds to a deformation of the Heisenberg–Weyl algebra with classical standard r-matrix, $r=\hat{z}\mathcal{C}\wedge\mathcal{B}$.

(2.2.2.4)

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=\mathcal{B} \qquad [\mathcal{B},\mathcal{C}]=0. \tag{4.14}$$

This is a non-coboundary deformation of the Heisenberg–Weyl algebra.

4.3. Case 3: $\rho=-1$

The classification of the quantum algebras corresponding to the case $\rho=-1$ can be performed by considering the transformation:

$$\mathcal{A}=\alpha A \qquad \mathcal{B}=\beta B+\delta\frac{\sinh(zA)}{z} \qquad \mathcal{C}=\nu C+\eta\frac{\sinh(zA)}{z} \qquad \hat{z}=\alpha^{-1}z \tag{4.15}$$

where $\alpha,\beta,\delta,\nu,\eta$ are complex numbers. So, we obtain the following classes of quantum algebras:

(3.1) $c_2\neq 0$

$$[\mathcal{A},\mathcal{B}]=\mathcal{B} \qquad [\mathcal{A},\mathcal{C}]=\mathcal{C} \qquad [\mathcal{B},\mathcal{C}]=0. \tag{4.16}$$

We have a deformation of the dilation algebra with no classical r-matrix.

(3.2) $c_2=0$:

(3.2.1)

$$\begin{aligned}&[\mathcal{A},\mathcal{B}]=-\sinh(\hat{z}\mathcal{A})/\hat{z}\\&[\mathcal{A},\mathcal{C}]=\sinh(\hat{z}\mathcal{A})/\hat{z}\\&[\mathcal{B},\mathcal{C}]=\mathcal{A}+(\mathcal{B}+\mathcal{C})\cosh(\hat{z}\mathcal{A}).\end{aligned} \tag{4.17}$$

Like in the previous case, this is a non-coboundary deformation.
Note that in the limit of $\hat{z}\to 0$ we get

$$[\mathcal{A},\mathcal{B}]=-\mathcal{A} \qquad [\mathcal{A},\mathcal{C}]=\mathcal{A} \qquad [\mathcal{B},\mathcal{C}]=\mathcal{A}+(\mathcal{B}+\mathcal{C})$$

that can be rewritten under a change ($\mathcal{B}+\mathcal{C}\to\mathcal{B}$) in the form

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=\mathcal{A} \qquad [\mathcal{B},\mathcal{C}]=\mathcal{A}+\mathcal{B}. \tag{4.18}$$

Therefore, we have obtained a quantum deformation of the Lie algebra (3.3).

(3.2.2) Other deformation of (3.3), now of non-standard type, is

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=\sinh(\hat{z}\mathcal{A})/\hat{z} \qquad [\mathcal{B},\mathcal{C}]=\mathcal{A}+\mathcal{B}\cosh(\hat{z}\mathcal{A}). \tag{4.19}$$

The r-matrix is $r=\hat{z}\mathcal{A}\wedge\mathcal{C}$.

(3.2.3)

$$\begin{aligned}&[\mathcal{A},\mathcal{B}]=-\sinh(\hat{z}\mathcal{A})/\hat{z}\\&[\mathcal{A},\mathcal{C}]=\sinh(\hat{z}\mathcal{A})/\hat{z}\\&[\mathcal{B},\mathcal{C}]=(\mathcal{B}+\mathcal{C})\cosh(\hat{z}\mathcal{A}).\end{aligned} \tag{4.20}$$

This is another non-coboundary deformation of the dilation algebra in two dimensions.

(3.2.4)

$$[\mathcal{A},\mathcal{B}]=0 \qquad [\mathcal{A},\mathcal{C}]=0 \qquad [\mathcal{B},\mathcal{C}]=\mathcal{A}. \tag{4.21}$$

It corresponds to a coboundary deformation of the Heisenberg–Weyl algebra with standard r-matrix, $r=\hat{z}\mathcal{B}\wedge\mathcal{C}$.

(3.2.5)

$$[\mathcal{A},\mathcal{B}]=-\sinh(\hat{z}\mathcal{A})/\hat{z} \qquad [\mathcal{A},\mathcal{C}]=0 \qquad [\mathcal{B},\mathcal{C}]=\mathcal{C}\cosh(\hat{z}\mathcal{A}). \tag{4.22}$$

We have another deformation of the dilation algebra in two dimensions. In this case the classical r-matrix is a non-standard one, $r=\hat{z}\mathcal{A}\wedge\mathcal{B}$.

5. Conclusions and remarks

As a result, we have shown that the quantization method presented here can be simultaneously applied and successfully solved for a multiparameter family of 3D Lie bialgebras that share some structural properties. In particular, note that the parameter ρ is a complex one, but we have obtained many interesting 'isolated' solutions for $\rho=\pm1$. On the other hand, the comparison with the complete classification of 3D real Lie bialgebras given in [1] can be worked out by considering the well-known isomorphisms among the complexified versions of 3D real Lie algebras.

It can be shown that all the quantum algebras that we have obtained in section 4 are quantizations of the complexifications of the dual version of the Lie bialgebras given in [1]. In order to find out the correspondence explicitly in table III of [1], we have to identify a given algebra $\mathfrak{a}$ with the complex version of the *dual* Lie algebra $\mathfrak{g}^*$ and, consequently, the dual of the cocommutator η will have to be isomorphic to one of the algebras $\mathfrak{g}$ in the first row of such a table. By proceeding in this way we find the following correspondences, where we firstly write the Lie bialgebras ($\mathfrak{g}^*$, $\mathfrak{g}\equiv\eta^*$) as labelled in table III of [1] and, afterwards, the corresponding quantum algebra according to our classification:

$5\to$ 1.2.2) $\quad 6\to$ 1.2.1) $\quad 7\to$ 1.1.1)

(1) $\to$ 2.1.1) $\quad$ (2), (4) $\to$ 2.1.1) $\quad$ (3) $\to$ 2.2.1) $\quad 9\to$ 2.1.2) $\quad 11, 11'\to$ 2.2.2.2)

$10\to$ 2.2.2.4) $\quad 5_{\rho=1}\to$ 2.2.2.3) $\quad 6_{\rho=1}\to$ 2.2.2.1) $\quad 7_{\rho=1}\to$ 2.1.2)$_{\rho=1}$

$5'\to$ 3.2.4) $\quad 8\to$ 3.2.1) $\quad$ (14) $\to$ 3.2.2) $\quad$ (11) $\to$ 3.2.3) $\quad 6_{\rho=-1}\to$ 3.2.5)$_{\rho=-1}$

$7_{\rho=-1}\to$ 3.1)$_{\rho=-1}$.

We can realize that our choice (3.4) for the cocommutator implies that we have been able to obtain the quantizations for the full set of dual Lie bialgebras ($\mathfrak{g}^*$, $\mathfrak{g}\equiv\eta$) of [1] such that $\eta^*\equiv\mathfrak{r}_3(\rho)$ for all values of ρ. In fact, the ρ parameter in (3.4) can be identified with the one appearing in Gomez's classification.

Finally, we would like to comment that the quantization procedure presented here can be, in principle, applied to higher dimensions with no need of further hypotheses, although it will certainly be cumbersome but not impossible with the help of a computer. In this respect, we recall that the complete classification of Lie bialgebra structures is only known for a restricted number of higher dimensional Lie algebras. Moreover, in a completely general situation, one would expect that our 'order by order' quantization procedure will have to be simultaneously solved for both the quantum coproduct and the deformed commutation rules, whilst in the 3D case presented here the full quantum coproduct (3.6) has been integrated independently.

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